Spacecraft Mass

14.1 Introduction

The design of a spacecraft is largely determined by the available mass budget. The following mass characteristics play an important role:

- mass of spacecraft that is to be launched
- the position of the centre of gravity or centre of mass
- the second moment of mass (moment of inertia, MOI)

The mass characteristics are always important. The level of importance of the other characteristics depends on the mission.

The calculation of the mass characteristics will be discussed in detail in the following sections.

The position of the centre of gravity is often of crucial importance because it is of great importance during launch. During lateral acceleration the spacecraft exerts a bending moment on the payload adapter. This bending moment depends on the mass of the spacecraft and the position of the centre of gravity. The launch authority specifies the permissible bending moment, from which the position of the centre of gravity can be calculated. The offset d with respect to the launch axis is also specified, for example $d \leq 3$ mm. The position of the centre of gravity is also important for the attitude control system (AOCS subsystem).

The second moments of mass are also important for the attitude control system.

The mutual relation between the mass moments of inertia is important for spinning satellites.

Satellites are classified according to weight (definition by the Surrey Satellite Technology Ltd (SSTL) , University of Surrey, Guildford, UK):

- Large spacecraft> 1000 kg
- Medium sized spacecraft 500–1000 kg
- Mini spacecraft 100–500 kg
- Micro spacecraft 10–100 kg

- Nano spacecraft 1–10 kg
- Pico spacecraft < 1 kg

Spacecraft that belong to the category 500–1000 kg are often referred to as "Small spacecraft" or "Smallsats" and are associated with cheap spacecraft that can be produced quickly ("Faster, Better, Smaller and Cheaper").

The "US Advanced Research Projects Agency" refers to the "Smallsats" as "Lightsats, while the "US Naval Command" refers to the "Smallsats" as "SPIN-Sat's" (Single Purpose Inexpensive Satellite Systems).

The principle of a mass budget is a method of bookkeeping: each subsystem is designed according to the goals set by the mass budget so that the mass can be monitored during the spacecraft project. In order to do this, a detailed list is used to record the mass of all the components of the spacecraft. At the beginning of the project the list consists mainly of the calculated masses, some multiplied with uncertainty factors. During the proposal phase 35% contingency is taken into account, at the preliminary design review (PDR) 15% contingency and at the critical design review 5% contingency is included in the mass analysis. As the design of the spacecraft is more and more frozen the uncertainties decrease and when hardware becomes available the measured weights can be included in the list.

In the following Table 14.1 the relative masses of the subsystems are given in terms of percentages. The percentages are averaged out over a number of spacecraft. This only concerns the dry mass of the spacecraft. The balance weight of approximately 1% is included. The fuel required to position a spacecraft in a geostationary orbit (GEO) and the fuel needed for the attitude control system can double the total weight with respect to the dry mass.

Table 14.1 Mass allocation budgets

	Mass budget (dry mass)	
Subsystem	**3-axis stabilised spacecraft (%)**	**Spinning spacecraft**
Structure	18	21
Propulsion (AKM+RCS)	12	11
AOCS	7	5
Power	23	24
TT&C	4	5
Thermal	4	5
Payload (incl. antennae)	28	25
Wiring	4	4
Total	100	100

14.2 Structure Mass

The structure of a spacecraft makes up in average about 20–21% of the total dry mass [Saleh 2002]. Mark Williamson [Williamson 1990] gives a mathematical relation for the percentage of the total mass that the structure mass, depending on the total dry mass, for a spacecraft stabilised along three axes as well as a spinning spacecraft:

The percentage of the structure mass with respect to the total dry mass of a spacecraft stabilised along three axes is:

$$p = -16\log(G) + 60\,, \tag{14.1}$$

with G the total dry mass of the spacecraft and p the percentage of structural mass of the total spacecraft mass.

The percentage of the structure mass with respect to the total dry mass of a spinning spacecraft is:

$$p = 16\log(G) - 60\,. \tag{14.2}$$

Below $G = 500$ kg the structural weight of a spinning spacecraft is lower than that of a spacecraft stabilised along three axes. Above 500 kg the structural weight of a spacecraft stabilised along three axes is lower than that of a spinning spacecraft.

In his book, B.N. Agrawal also gives an estimate for the structural weight, but with respect to the entire spacecraft weight (dry mass + fuel mass). For a spinning spacecraft that is 8.7% and for a spacecraft stabilised along three axes that is 9.7%.

14.3 Total Mass Calculation

The total mass of the spacecraft is of great importance and must therefore be calculated accurately. The following mass characteristics are important for the design of a spacecraft structure:

- The 6 x 6 mass matrix as rigid body
- The centre of gravity of the spacecraft: The first moments of mass are zero (kgm)
- The principal second moments of mass and the associated axes of inertia: The cross second moments of mass are zero (kgm)

14.3.1 Mass Matrix

The 6*x*6 diagonal mass matrix of a rigid body (instrument, box, etc.) $[M_{\mathrm{RB}}]$ is generally presented with respect to the centre of mass and in an orientation (local coordinate system) of the principle second moments of mass. The diagonal mass matrix $[M_{\mathrm{RB}}]$ is

$$[M_{\mathrm{RB}}] = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\bar{x}} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\bar{y}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\bar{z}} \end{bmatrix}. \tag{14.3}$$

This mass matrix $[M_{\mathrm{RB}}]$ will be transformed in a mass matrix with respect to the reference coordinate system x, y and z. The local coordinate system of the rigid body $\bar{x}$, $\bar{y}$ and $\bar{z}$ will be expressed in the reference coordinate system

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = [T]\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \tag{14.4}$$

where the rectangular matrix $[T]$ is an orthogonal transformation matrix with orthonormal columns and has special properties, $[T]^T[T] = [I]$ and $[T]^T = [T]^{-1}$, the left-inverse matrix [Strang 1988]. The reference coordinate system will be expressed in the local coordinate system of the rigid body

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [T]^{-1}\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = [T]^T\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix}. \tag{14.5}$$

A point P with coordinates (x, y, z) will be represented by a vector. The vector from the origin to the point P can be written as

$$\{P\} = xe_1 + ye_1 + ze_3 = x\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + y\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + z\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \tag{14.6}$$

with $e_j, j = 1, 2, 3$ as the unit orthonormal vectors representing the reference coordinate system.

The point P may also be expressed in the local coordinate system, however, the origin is the same.

$$\{P\} = \bar{x}\bar{e}_1 + \bar{y}\bar{e}_1 + \bar{z}\bar{e}_3. \tag{14.7}$$

The unit vectors $\bar{e}_i,\ i = 1, 2, 3$ may be expressed in the reference coordinate system.

$$\bar{e}_1 = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix},\ \bar{e}_2 = \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \text{ and } \bar{e}_3 = \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix}. \tag{14.8}$$

We know that the length of the vectors $|\bar{e}_1| = |\bar{e}_2| = |\bar{e}_3| = 1$, the inner-product of vectors $\bar{e}_i.\bar{e}_j = \delta_{ij}$, and the cross-product of vectors $\bar{e}_1 x \bar{e}_2 = \bar{e}_3$, $\bar{e}_2 x \bar{e}_3 = \bar{e}_1$ and $\bar{e}_3 x \bar{e}_1 = \bar{e}_2$

The coordinates of point P are written as follows (see (14.7))

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = [T]^T \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix}. \tag{14.9}$$

It is noticed that

$$x_i = \bar{e}_i.e_1,\ y_i = \bar{e}_i.e_2 \text{ and } z_i = \bar{e}_i.e_3. \tag{14.10}$$

The inner product of two vectors $x_i = \bar{e}_i.e_1$ is defined as

$$x_i = \bar{e}_i.e_1 = \lfloor\, x_i\ y_i\ z_i \,\rfloor \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}. \tag{14.11}$$

The translational velocities and angle velocities of a rigid body about the centre of mass of the rigid body can be defined for the local coordinate system $\lfloor \dot{\bar{x}} \rfloor = \lfloor\, \dot{\bar{u}}\ \dot{\bar{v}}\ \dot{\bar{w}}\ \dot{\bar{\varphi}}_{\bar{x}}\ \dot{\bar{\varphi}}_{\bar{y}}\ \dot{\bar{\varphi}}_{\bar{z}} \,\rfloor$ and for the reference coordinate system $\lfloor \dot{x} \rfloor = \lfloor\, \dot{u}\ \dot{v}\ \dot{w}\ \dot{\varphi}_x\ \dot{\varphi}_y\ \dot{\varphi}_z \,\rfloor$.

The relation between the velocities in the local and reference coordinate system is

$$\{\dot{\bar{x}}\} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \{\dot{x}\} = [T_{\mathrm{LR}}]\{\dot{x}\} \text{ and } \{\dot{x}\} = [T_{\mathrm{LR}}]^T\{\dot{\bar{x}}\}. \tag{14.12}$$

The kinetic energy is a scalar, thus

$$T = \frac{1}{2}\{\dot{\bar{x}}\}^T[M_{\mathrm{RB}}]\{\dot{\bar{x}}\} = \frac{1}{2}\{\dot{x}\}^T[M_{\mathrm{RB,ref}}]\{\dot{x}\}. \tag{14.13}$$

It can be seen that after the transformation from the local coordinate system to the reference coordinate system, the mass matrix of the rigid body $[M_{\mathrm{RB,ref}}]$ becomes

$$[M_{\mathrm{RB,ref}}] = [T_{\mathrm{LR}}][M_{\mathrm{RB}}][T_{\mathrm{LR}}]^T. \tag{14.14}$$

Example

The local coordinate is with respect to the reference coordinate system rotated 45° about the x-axis. The transformation matrix T will be

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

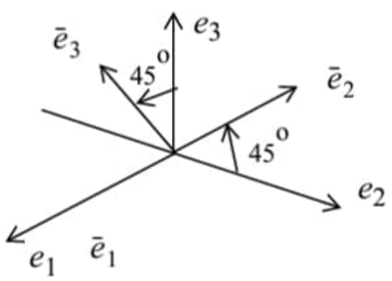

Fig. 14.1 Coordinate transformation

The mass matrix of the rigid body in the coordinate system is given by

$$[M_{\mathrm{RB}}] = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

The mass matrix with respect to the reference coordinate system can be calculated using (14.14)

$$[M_{\mathrm{RB,ref}}] = [T_{\mathrm{LR}}][M_{\mathrm{RB}}][T_{\mathrm{LR}}]^T = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 2.5 \end{bmatrix}.$$

End example

The mass matrix as rigid body with respect to an arbitrary point and coordinate system or for example the origin of the reference coordinate system is given by:

$$[M_{\mathrm{ref}}] = \begin{bmatrix} m_{xx} & 0 & 0 & 0 & S_{xy} & S_{xz} \\ 0 & m_{yy} & 0 & S_{yx} & 0 & S_{yz} \\ 0 & 0 & m_{zz} & S_{zx} & S_{zy} & 0 \\ 0 & S_{yx} & S_{zx} & I_{xx} & I_{xy} & I_{xz} \\ S_{xy} & 0 & S_{zy} & I_{yx} & I_{yy} & I_{yz} \\ S_{xz} & S_{yz} & 0 & I_{zx} & I_{zy} & I_{zz} \end{bmatrix}, \tag{14.15}$$

with m_{ii} the mass as rigid body in the i-direction, S_{ij} the first moment of mass in the *i*-direction about the *j*-direction and I_{ij} the second moment of mass in the *i*-direction about the *j*-direction. Normally $m_{xx} = m_{yy} = m_{zz} = m$.

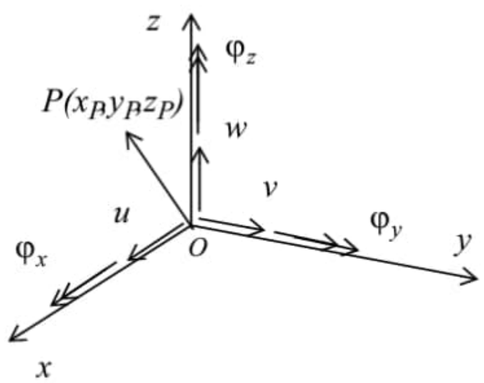

Fig. 14.2 Transformation of displacement vector

The coordinate system of point P is in the reference coordinate system. The displacement vector $\lfloor x_P \rfloor = \lfloor u \; v \; w \; \varphi_x \; \varphi_y \; \varphi_z \rfloor$ at P will be expressed in the displacement vector $\lfloor x_O \rfloor$ as follows

$$\begin{Bmatrix} u_P \\ v_P \\ w_P \\ \varphi_{x,P} \\ \varphi_{y,P} \\ \varphi_{z,P} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z_p & -y_p \\ 0 & 1 & 0 & -z_p & 0 & x_p \\ 0 & 0 & 1 & y_p & -x_p & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_O \\ v_O \\ w_O \\ \varphi_{x,O} \\ \varphi_{y,O} \\ \varphi_{z,O} \end{Bmatrix} = [T_P] \begin{Bmatrix} u_O \\ v_O \\ w_O \\ \varphi_{x,O} \\ \varphi_{y,O} \\ \varphi_{z,O} \end{Bmatrix}, \tag{14.16}$$

where the matrix $[T_P]$ is called the geometric matrix.

The inverse transformation of the geometric matrix $[T_P]$ is

$$\begin{Bmatrix} u_O \\ v_O \\ w_O \\ \varphi_{x,O} \\ \varphi_{y,O} \\ \varphi_{z,O} \end{Bmatrix} = [T_P]^{-1} \begin{Bmatrix} u_P \\ v_P \\ w_P \\ \varphi_{x,P} \\ \varphi_{y,P} \\ \varphi_{z,P} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -z_p & y_p \\ 0 & 1 & 0 & z_p & 0 & -x_p \\ 0 & 0 & 1 & -y_p & x_p & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_P \\ v_P \\ w_P \\ \varphi_{x,P} \\ \varphi_{y,P} \\ \varphi_{z,P} \end{Bmatrix}. \tag{14.17}$$

The kinetic energy of the general rigid body at point P is given by

$$T = \frac{1}{2}\{\dot{x}_P\}^T[M_{\mathrm{RB},P}]\{\dot{x}_P\} = \frac{1}{2}\{\dot{x}_O\}^T[T_P]^T[M_{\mathrm{RB},P}][T_P]^1\{\dot{x}_O\}, \tag{14.18}$$

or

$$T = \frac{1}{2}\{\dot{x}_O\}^T[T_P]^T[M_{\mathrm{RB},P}][T_P]^1\{\dot{x}_O\} = \frac{1}{2}\{\dot{x}_O\}^T[M_{\mathrm{RB},O,P}]\{\dot{x}_O\}. \tag{14.19}$$

Thus the transformed mass matrix in the origin of the reference coordinate system becomes

$$[M_{\mathrm{RB},O,P}] = [T_P]^T[M_{\mathrm{RB},P}][T_P]^1, \tag{14.20}$$

and the inverse transformation is

$$[M_{\mathrm{RB},P}] = [T_P]^T[M_{\mathrm{RB},O,P}][T_P]. \tag{14.21}$$

This will be repeated for all N_P mass systems, thus

$$[M_{\mathrm{RB},O}] = \sum_{P=1}^{N_P}[M_{\mathrm{RB},O,P}]. \tag{14.22}$$

14.3.2 Mass matrix with respect to the centre of mass

The mass matrix of a spacecraft is calculated with respect to the centre of mass and with axes parallel to the reference coordinate system.

$$[M_{\mathrm{RB,CG}}] = [T_{\mathrm{CG}}]^{-T}[M_{\mathrm{RB},O}][T_{CG}]^{-1}. \tag{14.23}$$

The transformation matrix $[T_{CG}]$ is defined as follows:

$$[T_{\mathrm{CG}}] = \begin{bmatrix} 1 & 0 & 0 & 0 & z_{CG} & -y_{CG} \\ 0 & 1 & 0 & -z_{CG} & 0 & x_{CG} \\ 0 & 0 & 1 & y_{CG} & -x_{CG} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14.24}$$

This is in accordance with (14.16). The inverse of $[T_{CG}]$ is in accordance with (14.17).

14.3.3 Centre of mass

If the mass matrix is given with respect to the origin of the coordinate system, with respect to the centre of gravity, the first order moments of mass S_{ij}, $i = 1, 2, 3$ $j = 1, 2, 3$ will vanish.

If (14.22) is substituted in the upper-right position of the first moments of mass we find the following values

$$[M_{\mathrm{RB,\,CG}}] = \begin{bmatrix} \ldots & 0 & mz_{\mathrm{CG}} + S_{xy} & -my_{\mathrm{CG}} + S_{xz} \\ \ldots & -mz_{\mathrm{CG}} + S_{yx} & 0 & mx_{\mathrm{CG}} + S_{yz} \\ \ldots & my_{\mathrm{CG}} + S_{zx} & -mx_{\mathrm{CG}} + S_{zy} & 0 \\ \ldots & . & . & . \\ \ldots & . & . & . \\ \ldots & . & . & . \end{bmatrix}. \tag{14.25}$$

The first moments of mass must vanish at the centre of mass.
The coordinates of the centre of gravity with respect to the reference coordinate system are:

$$x_{CG} = \frac{-S_{yz}}{m_{yy}} = \frac{S_{zy}}{m_{zz}}, \tag{14.26}$$

$$y_{CG} = \frac{-S_{zx}}{m_{zz}} = \frac{S_{xz}}{m_{xx}}, \tag{14.27}$$

$$z_{CG} = \frac{-S_{xy}}{m_{xx}} = \frac{S_{yx}}{m_{yy}}. \tag{14.28}$$

14.3.4 Second Moments of Mass

The symmetric second moments of mass $[I_{CG}]$ of the spacecraft with respect to the centre of gravity and with axes parallel to the reference coordinate system are:

$$[I_{CG}] = \begin{bmatrix} \bar{I}_{xx} & \bar{I}_{xy} & \bar{I}_{xz} \\ \bar{I}_{zx} & \bar{I}_{yy} & \bar{I}_{yz} \\ \bar{I}_{zx} & \bar{I}_{zy} & \bar{I}_{zz} \end{bmatrix}. \tag{14.29}$$

The principal second moments of mass $[I_{\text{principal}}]$ with respect to the centre of gravity and the associated principal axes of inertia $[Q]$ are:

$$[I_{\text{principal}}] = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, [Q] = \begin{bmatrix} q_{11} & q_{12} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{11} & q_{23} & q_{33} \end{bmatrix}. \tag{14.30}$$

The theory in this subsection will be illustrated by an example.

Example

There are three identical mass point $N_P = 3$ with the following mass matrix

$$[M_{RB,P}] = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The coordinates of the mass points are with respect to the origin of the global coordinate system.

- $P_1 \quad \lfloor x,y,z \rfloor = \lfloor 0.5,0,0 \rfloor$
- $P_2 \quad \lfloor x,y,z \rfloor = \lfloor 0,0.5,0 \rfloor$
- $P_3 \quad \lfloor x,y,z \rfloor = \lfloor 0,0,0.5 \rfloor$

The total mass matrix at the origin becomes

$$[M_{\mathrm{RB},O}] = \begin{bmatrix} 30 & 0 & 0 & 0 & 0-5 & 5 \\ 0 & 30 & 0 & 5 & 0 & -5 \\ 0 & 0 & 30 & -5 & 5 & 0 \\ 0 & 5 & -5 & 8 & 0 & 0 \\ -5 & 0 & 5 & 0 & 8 & 0 \\ 5 & -5 & 0 & 0 & 0 & 8 \end{bmatrix}.$$

The centre of mass is given by $P_{CG} \quad \lfloor x,y,z \rfloor_{CG} = \lfloor 0.1667, 0.1667, 0.1667 \rfloor$. The mass matrix with respect to the centre of gravity becomes

$$[M_{\mathrm{RB,CG}}] = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.3333 & 0.8333 & 0.8333 \\ 0 & 0 & 0 & 0.8333 & 6.3333 & 0.8333 \\ 0 & 0 & 0 & 0.8333 & 0.8333 & 6.3333 \end{bmatrix}.$$

The principle second moments of mass is now

$$[I_{\mathrm{principle}}] = \begin{bmatrix} 5.5 & 0 & 0 \\ 0 & 5.5 & 0 \\ 0 & 0 & 8.0 \end{bmatrix},$$

and the associated eigenvectors $[Q]$

$$[Q] = \begin{bmatrix} 0.7071 & 0.4082 & 0.5774 \\ -0.7071 & 0.4082 & 0.5774 \\ 0 & -0.8165 & 0.5774 \end{bmatrix}.$$

End of example

14.3.5 Finite Element Model Mass Matrix

The finite element mass matrix will be calculated, in general, in a local coordinate system. The element mass matrix will be transformed in the global system and added to the global mass matrix $[M]$. The transformation of the global mass matrix $[M]$ into a 6x6 rigid body mass matrix can be done using the rigid body modes with respect to the origin of the global coordinate system. The stiffness matrix $[K]$ and the geometric matrices are used to calculate the rigid body modes.

Afterwards the centre of mass and the principal second moments of mass can be obtained using (14.25) and (14.29).

The rigid body modes can be calculated as follows.

Rigid-Body Modes

If the linear dynamic system is not constrained the system can move as a rigid body. This means that during the movement as a rigid body no elastic forces will occur in the dynamic system. If this is the case, the stiffness matrix $[K]$ is singular (semi-positive-definite). In general, there are 6 possible motions as rigid-body; three translations and three rotations. The six rigid-body modes can be calculated very easily using the stiffness matrix $[K]$. The free-free system (with n degrees of freedom) is constrained at one node (i.e. the origin of the global system) with 6 degrees of freedom; three translations and three rotations. The set of degrees of freedom is called the *R*-set. The other elastic degrees of freedom are placed in the *E*-set, such that $n = R + E$. The constrained R-set is determinate, so no strains will be introduced in the elastic system. The R-set (origin) consists of 6 unit displacement and rotations and those will be enforced on the system. The rigid body motion can be calculated with the following equation

$$\begin{bmatrix} K_{EE} & K_{ER} \\ K_{RE} & K_{RR} \end{bmatrix} \begin{Bmatrix} \Phi_{R,E} \\ I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{14.31}$$

where $[I]$ is the identity matrix and $[\Phi_{R,E}]$ the *E*-set part of the rigid-body motion.

From the first equation of (14.31) the *E*-set part of the rigid-body mode can be solved

$$[\Phi_{R,E}] = -[K_{EE}]^{-1}[K_{ER}]. \tag{14.32}$$

The complete matrix of the 6 rigid body modes becomes

$$[\Phi_R] = \begin{bmatrix} -[K_{EE}]^{-1}[K_{ER}] \\ I \end{bmatrix}. \tag{14.33}$$

The rigid-body mode may be either extracted from the eigenvalue problem or calculated by partitioning the stiffness matrix in *E*-set and *R*-set submatrices. Using the geometric information of the nodes with coordinates (x,y,z) there is another way to calculate the rigid-body mode.

The geometric matrix

The geometric matrix of a node is obtained by translations along the x-, y- and z-axis and rotations about the x-, y- and z-axis. In fact, in the geometric matrix the motion of point P with respect to origin O is given with matrix $[T_P]$ (see (14.16)).

The rigid-body mode is built up from the geometric matrices of all nodes with respect to the origin of the global coordinate system. There are six rigid-body modes. The rigid body motion with respect to the origin is a column assembly of the node geometric matrices

$$[\Phi_R] = \begin{bmatrix} T_1 \\ \cdot \\ \cdot \\ T_N \end{bmatrix}, \qquad (14.34)$$

where N is the number of nodes.

The 6x6 rigid body matrix with respect to the origin of the global system can be obtained by

$$[M_{\mathrm{RB},O}] = [\Phi_R]^T[M][\Phi_R]. \qquad (14.35)$$

Afterwards the centre of mass and the principle second moments of mass can be obtained using (14.25) through (14.29).

Modal Effective Mass

16.1 Introduction

The modal effective mass is a modal dynamic property of a structure associated with the modal characteristics; natural frequencies, mode shapes, generalised masses, and participation factors. The modal effective mass is a measure to classify the importance of a mode shape when a structure is excited by base acceleration (enforced acceleration). A high effective mass will lead to a high reaction force at the base, while mode shapes with low associated modal effective mass are barely excited by base acceleration and will give low reaction forces at the base. The effect of local modes is not well described with modal effective masses [Shunmugavel 1995, Witting 1996].

The modal effective mass matrix is a $6x6$ mass matrix. Within this matrix the coupling between translations and rotations, for a certain mode shape, can be traced.

The summation over all modal effective masses will result in the mass matrix as a rigid-body.

In this chapter the theory behind the principle of the modal effective mass matrix will be discussed and the way in which the modal effective mass matrix can be obtained. The theory will be illustrated with an example.

16.2 Enforced Acceleration

An SDOF system with a discrete mass m, a damper element c and a spring element k is placed on a moving base that is accelerated with an acceleration $\ddot{u}(t)$. The resulting displacement of the mass is $x(t)$. The natural (circular) frequency

$\omega_n = \sqrt{\frac{k}{m}}$, the critical damping constant $c_{crit} = 2\sqrt{km}$ and the damping ratio $\zeta = \frac{c}{c_{crit}}$ are introduced. The amplification factor is defined as $Q = \frac{1}{2\zeta}$.

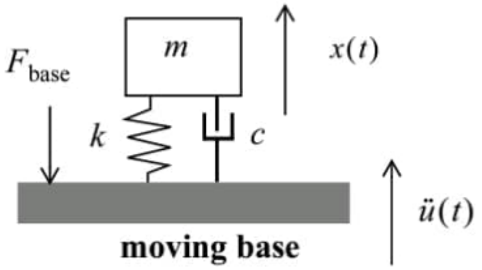

Fig. 16.1 Enforced acceleration of a damped SDOF system

A relative motion $z(t)$ is introduced, which is the displacement of the mass with respect to the base. The relative displacement is

$$z(t) = x(t) - u(t). \tag{16.1}$$

The equation of motion for the relative motion $z(t)$ is

$$\ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t) = -\ddot{u}(t). \tag{16.2}$$

The enforced acceleration of the SDOF system is transformed into an external force. The absolute displacement $x(t)$ can be calculated from

$$\ddot{x}(t) = \ddot{z}(t) + \ddot{u}(t) = -2\zeta\omega_n\dot{z}(t) - \omega_n^2 z(t). \tag{16.3}$$

The reaction force $F_{base}(t)$, due to the enforced acceleration $\ddot{u}(t)$, is a summation of the spring force and the damping force

$$F_{base}(t) = kz(t) + c\dot{z}(t) = -m\{\ddot{z}(t) + \ddot{u}(t)\} = -m\ddot{x}(t). \tag{16.4}$$

Assuming harmonic vibration we can write the enforced acceleration

$$\ddot{u}(t) = \ddot{U}(\omega)e^{j\omega t}, \tag{16.5}$$

and also the relative motion $z(t)$

$$z(t) = Z(\omega)e^{j\omega t},\ \dot{z}(t) = j\omega Z(\omega)e^{j\omega t} \text{ and } \ddot{z}(t) = -\omega^2 Z(\omega)e^{j\omega t} \tag{16.6}$$

and the absolute acceleration of the SDOF dynamic system is

$$\ddot{x}(t) = \ddot{X}(\omega)e^{j\omega t} = -\omega^2 X(\omega)e^{j\omega t}. \tag{16.7}$$

Equation (16.2) can be transformed in the frequency domain

$$[-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2]Z(\omega) = -\ddot{U}(\omega). \tag{16.8}$$

We are able to express the relative displacement $Z(\omega)$ in the enforced acceleration $\ddot{U}(\omega) = -\omega^2 U(\omega)$

$$Z(\omega) = \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right) U(\omega), \tag{16.9}$$

where $H(\omega) = \dfrac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\left(\frac{\omega}{\omega_n}\right)}$ is the frequency response function.

Using (16.3) we can write the absolute acceleration $\ddot{X}(\omega)$ as

$$\ddot{X}(\omega) = -\omega^2[Z(\omega) + U(\omega)] = -\omega^2\left[1 + \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right)\right]U(\omega), \tag{16.10}$$

or

$$\ddot{X}(\omega) = \left[1 + \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right)\right]\ddot{U}(\omega). \tag{16.11}$$

With the aid of (16.4) the reaction force at the base $F_{\mathrm{base}}(\omega)$ now becomes

$$F_{\mathrm{base}}(\omega) = m\ddot{X}(\omega) = m\left[1 + \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right)\right]\ddot{U}(\omega). \tag{16.12}$$

In this frame the mass m is the effective mass $M_{\mathrm{eff}} = m$. The reaction force $F_{\mathrm{base}}(\omega)$ is proportional to the effective mass M_{eff} and the base excitation $\ddot{U}(\omega)$ multiplied by the amplification $1 + \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right)$. Similar relations will be derived for multi-degrees of freedom (MDOF) dynamic systems.

When the excitation frequency is equal to the natural frequency of the SDOF $\omega = \omega_n$, the reaction force becomes

$$|F_{base}(\omega_n)| = \left|m\left[1 + \frac{1}{2j\zeta}\right]\ddot{U}(\omega_n)\right| \approx M_{\mathrm{eff}}Q\ddot{U}(\omega_n). \tag{16.13}$$

(16.12) can then be written in a dimensionless form

$$\frac{F_{\mathrm{base}}(\omega)}{m\ddot{U}(\omega)} = \left[1 + \left(\frac{\omega}{\omega_n}\right)^2 H\left(\frac{\omega}{\omega_n}\right)\right]. \tag{16.14}$$

16.3 Modal Effective Masses of an MDOF System

The undamped (matrix) equations of motion for a free-free elastic body can be written as

$$[M]\{\ddot{x}(t)\}+[K]\{x(t)\} = \{F(t)\}. \qquad (16.15)$$

The external or boundary degrees of freedom are denoted with the index j and the internal degrees of freedom with the index i. The structure will be excited at the boundary DOFs; 3 translations and 3 rotations.

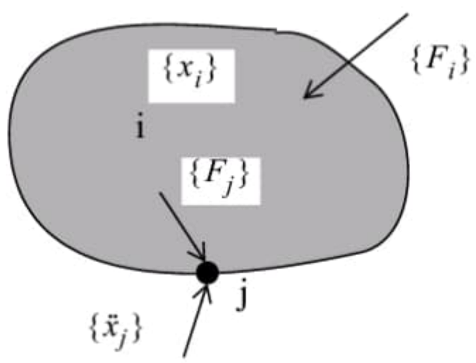

Fig. 16.2 Enforced structure

The number of boundary degrees of freedom is less than or equal to 6. The DOFs and forces are illustrated in Fig. 16.2. The matrix (16.15) may be partitioned as follows

$$\begin{bmatrix} M_{ii} & M_{ij} \\ M_{ji} & M_{jj} \end{bmatrix}\begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_j \end{Bmatrix}+\begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}\begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}. \qquad (16.16)$$

In [Craig 1968] it is proposed to depict the displacement vector $\{x(t)\}$ on a basis of 6 rigid-body modes $[\Phi_r]$ with $\{x_j\} = [I]$ and elastic mode shapes $[\Phi_p]$ with fixed external degrees of freedom $\{x_j\} = \{0\}$ calculated from the eigenvalue problem $([K_{ii}]-\langle\lambda_p\rangle[M_{ii}])[\Phi_{ii}] = [0]$. $\{x\}$ can be expressed as

$$\{x\} = [\Phi_r]\{x_j\}+[\Phi_p]\{\eta_p\} = [\Phi_r, \Phi_p]\begin{Bmatrix} x_j \\ \eta_p \end{Bmatrix} = [\Psi]\{X\}. \qquad (16.17)$$

The static modes can be obtained, assuming zero inertia effects, and $\{F_i\} = \{0\}$, and prescribe successively a unit displacement for the 6 boundary DOFs, thus $\{x_j\} = [I]$. (16.16) may be written as follows

$$\begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{16.18}$$

Enforced displacement $\{x_j\}$ will not introduce reaction forces in boundary DOFs.

From the first equation of (16.18) we find for $\{x_i\}$

$$[K_{ii}]\{x_i\} + [K_{ij}][x_j] = 0\,, \tag{16.19}$$

hence

$$\{x_i\} = -[K_{ii}]^{-1}[K_{ij}]\{x_j\}\,, \tag{16.20}$$

and therefore

$$[\Phi_{ij}] = -[K_{ii}]^{-1}[K_{ij}][I] = -[K_{ii}]^{-1}[K_{ij}]\,. \tag{16.21}$$

The static transformation now becomes

$$\{x\} = \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{bmatrix} \Phi_{ij} \\ I \end{bmatrix} \{x_j\} = [\Phi_r]\{x_j\}\,. \tag{16.22}$$

Using (16.18) it follows that

$$[K][\Phi_r] = \{0\}\,. \tag{16.23}$$

Assuming fixed external degrees of freedom $\{x_j\} = \{0\}$ and also assuming harmonic motions $x(t) = X(\omega)e^{j\omega t}$ the eigenvalue problem can be stated as

$$([K_{ii}] - \lambda_{k,p}[M_{ii}])\{X(\lambda_{k,p})\} = \{0\}, \tag{16.24}$$

or more generally as

$$([K_{ii}] - \langle\lambda_k\rangle[M_{ii}])[\Phi_{ip}] = \{0\}\,, \tag{16.25}$$

where λ_k the eigenvalue associated with the mode shape $\{\phi_{ip,k}\}$, $k = 1, 2, \ldots j$.

The internal degrees of freedom $\{x_i\}$ will be projected on the set of orthogonal mode shapes (modal matrix) $[\Phi_{ip}]$, thus

$$\{x_i\} = [\Phi_{ip}]\{\eta_p\}\,. \tag{16.26}$$

The modal transformation becomes

$$\{x\} = \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{bmatrix} \Phi_{ip} \\ 0 \end{bmatrix}\{\eta_p\} = [\Phi_p]\{\eta_p\}. \tag{16.27}$$

The Craig–Bampton (CB) transformation matrix $[\Psi]$ is

$$\{x\} = [\Phi_r, \Phi_p]\begin{bmatrix} x_j \\ \eta_p \end{bmatrix} = [\Psi]\{\chi\}, \tag{16.28}$$

where $[\Phi_r]$ the rigid body modes, $[\Phi_p]$ the modal matrix, $\{x_j\}$ the external or boundary degrees of freedom $(j \le 6)$ and $\{\eta_p\}$ the generalised coordinates. In general, the number of generalised coordinates p is much less than the total number of degrees of freedom $n = i + j$, $p \ll i$.

The CB transformation (16.28) will be substituted into (16.15) assuming equal potential and kinetic energies, hence

$$[\Psi]^T[M][\Psi]\{\ddot{\chi}\} + [\Psi]^T[K][\Psi]\{\chi\} = [\Psi]^T\{F(t)\} = \{f(t)\}, \tag{16.29}$$

further elaborated it is found

$$\begin{bmatrix} M_{rr} & M_{jp} \\ M_{pj} & \langle m_p \rangle \end{bmatrix}\begin{Bmatrix} \ddot{x}_j \\ \ddot{\eta}_p \end{Bmatrix} + \begin{bmatrix} \tilde{K}_{jj} & K_{jp} \\ K_{pj} & \langle k_p \rangle \end{bmatrix}\begin{Bmatrix} x_j \\ \eta_p \end{Bmatrix} = \begin{bmatrix} \Phi_{ij} & \Phi_p \\ I & 0 \end{bmatrix}^T\begin{Bmatrix} F_i \\ F_j \end{Bmatrix}, \tag{16.30}$$

with

- $[M_{rr}]$ the 6x6 rigid body mass matrix with respect to the boundary DOFs
- $[\tilde{K}_{jj}]$ the Guyan reduced stiffness matrix (j-set)
- $\langle m_p \rangle$ the diagonal matrix of generalised masses, $\langle m_p \rangle = [\Phi_p]^T[M][\Phi_p]$
- $\langle k_p \rangle$ the diagonal matrix of generalised stiffnesses,

$$\langle k_p \rangle = [\Phi_p]^T K[\Phi_p] = \langle \lambda_p \rangle \langle m_p \rangle = \langle \omega_p^2 \rangle \langle m_p \rangle$$

- $[K_{ip}] = [\Phi_{ij}]^T[K_{ii}][\Phi_p] + [K_{ji}][\Phi_p] = (-[K_{ij}]^T[K_{ii}]^{-1}[K_{ii}] + [K_{ji}])[\Phi_p] = [0]$
- $[K_{pi}] = [K_{ip}]^T = [0]$
- $[\tilde{K}_{jj}] = [\Phi_r]^T[K][\Phi_r] = [0]$

Thus (16.30) becomes

$$\begin{bmatrix} M_{rr} & L^T \\ L & \langle m_p \rangle \end{bmatrix} \begin{Bmatrix} \ddot{x}_j \\ \ddot{\eta}_p \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \langle m_p \lambda_p \rangle \end{bmatrix} \begin{Bmatrix} x_j \\ \eta_p \end{Bmatrix} = \begin{bmatrix} \Phi_{ij} & \Phi_p \\ I & 0 \end{bmatrix}^T \begin{Bmatrix} 0 \\ F_j \end{Bmatrix} = \begin{Bmatrix} F_j \\ 0 \end{Bmatrix}, \quad (16.31)$$

where $[M_{jp}]= [\Phi_r]^T[M][\Phi_p]= [L]^T$, $[L]^T$ is the matrix with the modal participation factors, $L_{kl} = \{\phi_{r,k}\}^T[M]\{\phi_{p,l}\}$, $k = 1,2,...,6$, $l = 1,2,...,p$.

The matrix of modal participation factors couples the rigid-body modes $[\Phi_r]$ with the elastic modes $[\Phi_p]$ and $\{F_i\} = \{0\}$ No internal loads are applied.

Introducing the modal damping ratios ζ_p (16.31) can be written as follows

$$\begin{bmatrix} M_{rr} & L^T \\ L & \langle m_p \rangle \end{bmatrix} \begin{Bmatrix} \ddot{x}_j \\ \ddot{\eta}_p \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \langle 2m_p\zeta_p\omega_p \rangle \end{bmatrix} \begin{Bmatrix} \dot{x}_j \\ \dot{\eta}_p \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \langle m_p \lambda_p \rangle \end{bmatrix} \begin{Bmatrix} x_j \\ \eta_p \end{Bmatrix} = \begin{Bmatrix} F_j \\ 0 \end{Bmatrix}$$

(16.32)

(16.32) can be divided into two equations

$$[M_{rr}]\{\ddot{x}_j\} + [L]^T\{\ddot{\eta}_p\} = \{F_j\}, \quad (16.33)$$

and

$$[L]\{\ddot{x}_j\} + \langle m_p \rangle \{\ddot{\eta}_p\} + \langle 2m_p\zeta_p\omega_p \rangle \{\dot{\eta}_p\} + \langle m_p\lambda_p \rangle \{\eta_p\} = \{0\}. \quad (16.34)$$

(16.33) and (16.34), when transformed in the frequency domain, give

$$[M_{rr}]\{\ddot{X}_j\} + [L]^T\{\ddot{\Pi}_p\} = \{F_j\}, \quad (16.35)$$

and

$$[L]\{\ddot{X}_j\} + \langle m_p \rangle \{\ddot{\Pi}_p\} + \langle 2m_p\zeta_p\omega_p \rangle \{\dot{\Pi}_p\} + \langle m_p\lambda_p \rangle \{\Pi_p\} = \{0\}, \quad (16.36)$$

with

- $x(t) = Xe^{j\omega t}$, $\ddot{X} = -\omega^2 X$
- $\eta(t) = \Pi e^{j\omega t}$, $\dot{\Pi} = j\omega\Pi$ and $\ddot{\Pi} = -\omega^2\Pi$
- $F(t) = \hat{F}e^{j\omega t}$

With (16.36) we express $\{\Pi_p\}$ in $\{X_j\}$

$$m_p[-\omega^2 + 2j\zeta_p\omega_p\omega + \omega_p^2]\Pi_p = -[L_p]\{\ddot{X}_j\}, \quad (16.37)$$

where $[L_k] = \{\phi_{p,k}\}^T[M][\Phi_r]$ is the 1x6 vector with modal participation factors and $L_{kj} = \{\phi_{p,k}\}^T[M]\{\Phi_{r,j}\}$ participation factor with $k = 1,2,...,p$ and $j = 1,2,...,6$.

Thus (16.37) becomes

$$\Pi_k = -\frac{[L_k]\{\ddot{X}_j\}}{m_k\omega_k^2}\left[\frac{1}{1-\left(\frac{\omega}{\omega_k}\right)^2 + 2j\zeta_k\frac{\omega}{\omega_k}}\right] = -\frac{[L_k]\{\ddot{X}_j\}}{m_k\omega_k^2}H_k\left(\frac{\omega}{\omega_k}\right). \tag{16.38}$$

(16.38) will be substituted into (16.35) giving

$$[M_{rr}]\{\ddot{X}_j\} + [L_1^T, ..., L_p^T]\left\{\left(\frac{\omega}{\omega_k}\right)^2\frac{[L_k]}{m_k}H_k\left(\frac{\omega}{\omega_k}\right)\right\}\{\ddot{X}_j\} = \{\hat{F}_j\}, k=1,2,...,p \tag{16.39}$$

$$\left[[M_{rr}] + \sum_{k=1}^{p}\frac{[L_k]^T[L_k]}{m_k}\left\{\left(\frac{\omega}{\omega_k}\right)^2 H_k\left(\frac{\omega}{\omega_k}\right)\right\}\right]\{\ddot{X}_j\} = \{\hat{F}_j\}. \tag{16.40}$$

We can prove that

$$[M_{rr}] = \sum_{k=1}^{p}\frac{[L_k]^T[L_k]}{m_k}, \tag{16.41}$$

because

$$[M_{rr}] = [\Phi_r]^T[M][\Phi_p]([\Phi_{ip}]^T[M_{ii}][\Phi_{ip}])^{-1}[\Phi_p]^T[M][\Phi_r] = [\Phi_r]^T[M][\Phi_r], \tag{16.42}$$

or

$$[M_{rr}] = [\Phi_{ij}^T, I][M]\begin{bmatrix}\Phi_{ip}\\ 0\end{bmatrix}([\Phi_{ip}]^T[M_{ii}][\Phi_{ip}])^{-1}[\Phi_{ip}^T, 0][M]\begin{bmatrix}\Phi_{ij}\\ 0\end{bmatrix} = [\Phi_r]^T[M][\Phi_r].$$

Assuming the inverse of $[\Phi_{ip}]$ exists. The modal effective mass $[M_{\text{eff},k}]$ is defined as follows

$$[M_{\text{eff},k}] = \frac{[L_k]^T[L_k]}{m_k}, \tag{16.43}$$

where $[L_k] = \{\phi_{p,k}\}^T[M][\Phi_r]$and $m_k = \{\phi_{p,k}\}^T[M]\{\phi_{p,k}\}$.

The summation over all modal effective masses $[M_{\text{eff},k}]$ will result in the rigid-body mass matrix $[M_{rr}]$ with respect to $\{x_j\}$. (16.41) becomes

$$[M_{rr}] = \sum_{k=1}^{p=i} [M_{\text{eff},k}] , \tag{16.44}$$

Therefore (16.40) can be written

$$\left[\sum_{k=1}^{p} [M_{\text{eff},k}]\left\{1 + \left(\frac{\omega}{\omega_k}\right)^2 H_k\left(\frac{\omega}{\omega_k}\right)\right\}\right]\{\ddot{X}_j\} = \{\hat{F}_j\} \tag{16.45}$$

(16.45) can be decomposed into modal reaction forces $\{F_{\text{base},k}\}$

$$\sum_{k=1}^{p} \{F_{\text{base},k}\} = \{\hat{F}_j\} , \tag{16.46}$$

with

$$\{F_{\text{base},k}\} = [M_{\text{eff},k}]\left\{1 + \left(\frac{\omega}{\omega_k}\right)^2 H_k\left(\frac{\omega}{\omega_k}\right)\right\}\{\ddot{X}_j\} . \tag{16.47}$$

(16.47) is very similar to (16.12).

Example

For the dynamic system, as illustrated in Fig. 16.3, the effective masses $[M_{\text{eff},k}]$ will be calculated. The parameters m (kg) and k (N/m) are, respectively, $m = 1$ and $k = 100000$. The set of internal DOFs is $\{x_i\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}^T$ and the boundary DOF is $\{x_j\} = \{x_8\}$.

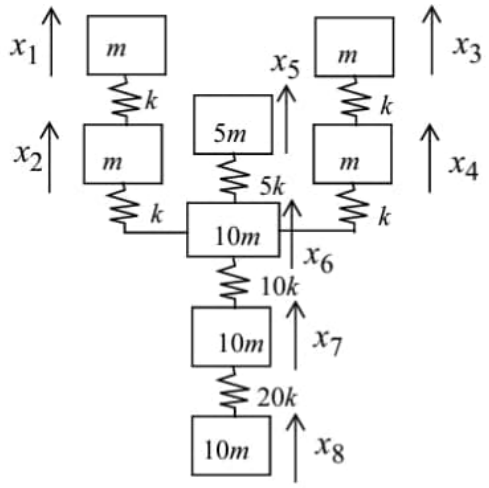

Fig. 16.3 8 DOFs dynamic system

The following procedure will be followed

1. $\{x_j\}$ will be assigned, $x_j = x_8$
2. Calculate the rigid-body modes $[\Phi_r] = \begin{bmatrix} -[K_{ii}]^{-1}[K_{ij}] \\ I \end{bmatrix}$, $x_j = x_8 = 1$
3. Fix the DOFs $\{x_j\}$, $x_j = x_8 = 0$
4. Calculate the natural frequencies and associated mode shapes $[\Phi_p]$, $x_j = x_8 = 0$
5. Assemble $[\Psi] = [\Phi_r \Phi_p]$
6. Calculate $[\Psi]^T[M][\Psi]$ and $[\Psi]^T[K][\Psi]$
7. Calculate the modal effective masses per mode $[M_{\text{eff},k}] = \dfrac{[L_k]^T[L_k]}{m_k}$
8. Calculate the summation of modal effective masses $[M_{rr}]= \sum_{k=1}^{p} [M_{\text{eff},k}]$

The rigid-body mode $\{\phi_r\}$, is with respect to $x_8 = x_j = 1$, and the natural frequencies and associated mode shapes are with respect to $x_8 = x_j = 0$ are

$$\{f_n\} = \begin{Bmatrix} 24.4522 \\ 31.1052 \\ 36.6716 \\ 64.4657 \\ 81.4344 \\ 82.0637 \\ 95.9164 \end{Bmatrix}, \; [\phi_r] = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$[\Phi_p] = \begin{bmatrix} 0.5347 & -0.6015 & -0.5781 & -0.3363 & -0.3717 & 0.3630 & -0.1343 \\ 0.4075 & -0.3717 & -0.2712 & 0.2155 & 0.6015 & -0.6022 & 0.3534 \\ 0.5347 & 0.6015 & -0.5781 & -0.3363 & 0.3717 & 0.3630 & -0.1343 \\ 0.4075 & 0.3717 & -0.2712 & 0.2155 & -0.6015 & -0.6022 & 0.3534 \\ 0.2407 & 0 & 0.3831 & -0.6458 & 0 & -0.0202 & 0.1681 \\ 0.1835 & 0 & 0.1797 & 0.4137 & 0 & 0.0336 & -0.4425 \\ 0.0664 & 0 & 0.0728 & 0.3044 & 0 & 0.0984 & 0.7001 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The mass matrix $[\Psi]^T[M][\Psi]$ and the stiffness matrix $[\Psi]^T[K][\Psi]$ become

$$[\Psi]^T[M][\Psi] = \begin{bmatrix} 39 & 5.5874 & 0 & 2.7421 & 3.7104 & 0 & 0.7400 & 3.8552 \\ 5.5874 & 1.5746 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 2.7421 & 0 & 0 & 1.9255 & 0 & 0 & 0 & 0 \\ 3.7104 & 0 & 0 & 0 & 5.0429 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0.7400 & 0 & 0 & 0 & 0 & 0 & 1.0989 & 0 \\ 3.8552 & 0 & 0 & 0 & 0 & 0 & 0 & 7.2863 \end{bmatrix}$$

$$[\Psi]^T[K][\Psi] = 10^6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0374 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0382 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1022 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8274 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2618 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2922 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.6464 \end{bmatrix}.$$

The results of the calculations are summarised in Table 16.1.

Table 16.1 Calculation of the modal effective masses

Mode shape #	**Natural frequency (Hz)**	**Modal participation factor $[L_k]^T$**	**Generalised masses $[m_k]$**	**Modal effective mass $[M_{\mathrm{eff},k}]$ (kg)**
1	24.5422	5.5874	1.5746	19.8271
2	31.1052	0.0000	1.0000	0.0000
3	36.6716	2.7421	1.9255	3.9048
4	64.4657	3.7104	5.0429	2.7300
5	81.4344	0.0000	1.0000	0.0000
6	82.0637	0.7400	1.0989	0.4983
7	95.9164	3.8552	7.2863	2.0398
	Total mass (without $m_8 = 10m$)			29.0000

The mass $m_8 = 10m$ (connected to DOF x_8) is eliminated because the elastic modes are with respect to x_8.

It appears that the modal effective mass of the first mode shape is already 68.37% of the total mass of 29 kg. The second and the fifth mode shapes have zero modal effective mass. Modes with zero modal effective mass cannot be excited in the case of enforced acceleration.

The absolute value of the normalised base force $\left|\frac{F_{\text{base}}(\omega)}{\ddot{X}(\omega)}\right|$ can be written as

$$\left|\frac{F_{\text{base}}(\omega)}{\ddot{X}(\omega)}\right| = \left|\sum_{k=1}^{7} [M_{em,k}]\left\{1 + \left(\frac{\omega}{\omega_k}\right)^2 H_k\left(\frac{\omega}{\omega_k}\right)\right\}\right|,$$

and the calculations are illustrated in Fig. 16.4.

End of example